MATH 140A Review: Inequalities and Absolute Values

1. Prove that if $0 < a_1 < a_2$ and $0 < b_1 < b_2$, then $a_1 \cdot b_1 < a_2 \cdot b_2$.

Solution:

Proof. Since $b_1 > 0$, then $a_1 \cdot b_1 < a_2 \cdot b_1$. Since $a_2 > 0$, then $a_2 \cdot b_1 < a_2 \cdot b_2$. Thus, $a_1 \cdot b_1 < a_2 \cdot b_2$. □

2. Let n be a natural number. Determine for what values of n the following holds

$$\left| \frac{n-1}{3n+1} - \frac{1}{3} \right| < \frac{1}{2020}.$$

Solution: We have that

$$\left| \frac{n-1}{3n+1} - \frac{1}{3} \right| < \frac{1}{2020} \tag{1}$$

if and only if

$$-\frac{1}{2020} + \frac{1}{3} < \frac{n-1}{3n+1} < \frac{1}{2020} + \frac{1}{3}.$$

By multiplying, (1) holds if and only if

$$-\frac{3n+1}{2020} + n + \frac{1}{3} < n - 1 < \frac{3n+1}{2020} + n + \frac{1}{3}.$$

By adding, (1) holds if and only if

$$-\frac{3n+1}{2020} < -\frac{4}{3} < \frac{3n+1}{2020}.$$

Hence, (1) holds if and only if

$$\frac{4}{3} < \frac{3n+1}{2020}.$$

That is, (1) holds if and only if

$$\left(\frac{4}{3} \cdot 2020 - 1 \right) /3 < n.$$

3. We say that a function f is *increasing* if $f(x) < f(y)$ when $x < y$. We say that f is *decreasing* if $-f$ is increasing. Let $f(x) = \frac{1}{x^p}$ for $x > 0$. Without taking the derivative, determine for what values of $p \in (-\infty, \infty)$ the function f is decreasing on $x > 0$ and increasing on $x > 0$.

Solution: Let $0 < x < y$. Since the function $f(z) = z^p$ is increasing for $p > 0$, then $x^p < y^p$. By multiplying both sides to the other side, we get that $f(y) < f(x)$. Hence, f is decreasing for $p > 0$. f is neither decreasing nor increasing for $p = 0$. Since the function $f(z) = z^p$ is decreasing for $p < 0$, then $0 < x < y$ implies that $x^p > y^p$. By multiplying both sides to the other side, we get that $f(y) > f(x)$. Hence, f is increasing for $p < 0$.